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On Ideals in H^∞ Whose Closures are Intersections of Maximal Ideals

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§1. Introduction

Let H^∞ be the Banach algebra of bounded analytic functions on the open unit disk D. We denote by $M(H^\infty)$ the set of non-zero multiplicative linear functionals of H^∞ endowed with the weak*-topology of the dual space of H^∞. Identifying a point in D with its point evaluation, we think as $D \subset M(H^\infty)$. For $\varphi \in M(H^\infty)$, put $Ker\,\varphi = \{f \in H^\infty; \varphi(f) = 0\}$. Then $Ker\,\varphi$ is a maximal ideal in H^∞, and for a maximal ideal I in H^∞ there exists $\psi \in M(H^\infty)$ such that $I = Ker\,\psi$. Usually $M(H^\infty)$ is called the maximal ideal space of H^∞. For $f \in H^\infty$, the function $\hat{f}(\varphi) = \varphi(f)$ on $M(H^\infty)$ is called the Gelfand transform of f. We identify f with $\hat{f}$, so that we think of H^∞ the closed subalgebra of continuous functions on $M(H^\infty)$. Let L^∞ be the Banach algebra of bounded measurable functions on ∂D. We denote by $M(L^\infty)$ the maximal ideal space of L^∞. We may think that $M(L^\infty) \subset M(H^\infty)$ and $M(L^\infty)$ is the Shilov boundary of H^∞, that is, the smallest closed subset of $M(H^\infty)$ on which every function in H^∞ attains its maximal modulus. A nice reference on this subject is [3].

For $f \in H^\infty$, there exists a radial limit $f(e^{i\theta})$ for almost everywhere. Let h be a bounded measurable function on ∂D such that $\int_0^{2\pi} \log|h|\, d\theta/2\pi > -\infty$. Put

$$f(z) = \exp\Big(\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log|h(e^{i\theta})|\, d\theta/2\pi \Big), \quad z \in D.$$

A function of this form is called outer, and $|f(e^{i\theta})| = |h(e^{i\theta})|$ almost everywhere. A function $u \in H^\infty$ is called inner if $|u(e^{i\theta})| = 1$ a.e. on ∂D. For a sequence $\{z_n\}_n$ in D with $\sum_{n=1}^\infty (1 - |z_n|) < \infty$, there corresponds a Blaschke product

$$b(z) = \prod_{n=1}^\infty \frac{-\overline{z}_n}{|z_n|} \frac{z - z_n}{1 - \overline{z}_n z}, \quad z \in D.$$

A Blaschke product is called interpolating if for every bounded sequence of complex numbers $\{a_n\}_n$ there exists $h \in H^\infty$ such that $h(z_n) = a_n$ for every n. For a non-negative bounded singular measure $\mu, \mu \neq 0$, on ∂D, let

$$\psi_\mu(z) = \exp\Big(- \int_{\partial D} \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu \Big), \quad z \in D.$$

Then ψ_μ is inner and called a singular function. It is well known that every function in H^∞ is factored as an inner function times an outer function, and an inner function is factored as a Blaschke product times a singular function.

For a subset E of $M(H^\infty)$, let $I(E) = \bigcap\{Ker\,\varphi; \varphi \in E\}$ be the intersection of maximal ideals associated with points in E. For $f \in H^\infty$, let $Z(f) = \{\varphi \in M(H^\infty); \varphi(f) = 0\}$ be the zero set of f. In this paper, we mean that an ideal is a non-zero proper ideal in H^∞. For an ideal I in H^∞, put $Z(I) = \bigcap\{Z(f); f \in I\}$, then $I \subset I(Z(I))$. An ideal I is called prime if for any $f, g \in H^\infty$ with $fg \in I$, then $f \in I$ or $g \in I$. There are many studies of prime ideals in H^∞, see [4, 14, 15, 16]. Recently, Gorkin and Mortini [6, Theorem 1] proved that a closed prime ideal I is an intersection of maximal ideals, that is, $I = I(Z(I))$. And they pointed out that if I is a (non-closed) prime ideal such that $Z(I) \cap M(L^\infty) = \emptyset$, then the closure of I is an intersection of maximal ideals, that is, $\overline{I} = I(Z(I))$.

Let E be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Let $J = J(E)$ be the ideal of H^∞ which consists of functions in H^∞ vanishing on some open subsets U of $M(H^\infty) \setminus D$ such that $E \subset U$. In [7, Theorem 4.2], Gorkin and Mortini also showed that $\overline{J} = I(Z(J))$. It is a very interesting problem to determine the class of ideals I satisfying $\overline{I} = I(Z(I))$. But it seems difficult to give a complete characterization of these ideals.

In Section 2, we introduce the following condition on ideals I in H^∞ to study ideals I satisfying $\overline{I} = I(Z(I))$. We prove that if an ideal I of H^∞ satisfyies condition (α), then $\overline{I} = I(Z(I))$. We also give some examples of ideals I satisfying condition (α).

In Section 3, we study an ideal $I(f)$ of H^∞ which is generated by a noninvertible outer function f. There exist noninvertible outer functions f and g satisfying $\overline{I(f)} = I(Z(I(f)))$ and $\overline{I(g)} \neq I(Z(I(g)))$. As an application of the theorem given in Section 2, we characterize noninvertible outer functions f satisfying $\overline{I(f)} = I(Z(I(f)))$.

(α) For any $0 < \sigma < 1$ and a subset E of D such that $Z(I) \cap cl\,E = \emptyset$, there exists $h \in I$ such that $\|h\|_\infty \leq 1$ and $|h| \geq \sigma$ on E, where $cl\,E$ is the weak*-closure of E in $M(H^\infty)$.

2. Closure of ideals

We introduce the following condition on ideals I in H^∞.

(α) For any $0 < \sigma < 1$ and a subset E of D such that $Z(I) \cap cl\,E = \emptyset$, there exists $h \in I$ such that $\|h\|_\infty \leq 1$ and $|h| \geq \sigma$ on E, where $cl\,E$ is the weak*-closure of E in $M(H^\infty)$.

The main theorem of this paper is the following.

THEOREM 2.1. *Let I be an ideal in H^∞ satisfying condition (α). Then $\overline{I} = I(Z(I))$.*

Generally the converse of Theorem 2.1 does not hold, but it holds for some ideals. Let G be the set of point φ in $M(H^\infty)$ such that $\varphi(b) = 0$ for some interpolating Blaschke product b. By Hoffman's work [11], G is an open subset of $M(H^\infty)$ and for each $\varphi \in G$ there exists a continuous one to one map L_φ from D into $M(H^\infty)$ such that $L_\varphi(0) = \varphi$ and $f \circ L_\varphi \in H^\infty$ for every $f \in H^\infty$. Put $P(\varphi) = L_\varphi(D)$, and this set is called the Gleason part containing φ. Then we have

PROPOSITION 2.1. *Let I be an ideal in H^∞ such that $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$. Then $\overline{I} = I(Z(I))$ if and only if I satisfies condition (α).*

By the proof of Theorem 2.1 and Proposition 2.1, we have

COROLLARY 2.1. *Let I be an ideal in H^∞ algebraically generated by countable functions. Suppose that $P(\varphi) \subset Z(I)$ for every $\varphi \in Z(I) \cap G$. Then $I(Z(I))$ is a closed ideal generated by countable functions.*

Examples of ideals satisfying condition (α) are given in the following.

PROPOSITION 2.2. *The following ideals I in H^∞ satisfy condition (α).*

(i) *I is a prime ideal in H^∞ which does not contain any interpolating Blaschke product.*

(ii) *Let f be a function in H^∞ which does not vanish on D. Let I be the ideal in H^∞ algebraically generated by functions $f^{1/n}, n = 1, 2, \ldots$.*

(iii) *Let E be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Let I be the ideal of functions in H^∞ which vanish on some open subsets U of $M(H^\infty) \setminus D$ such that $E \subset U$.*

(iv) *Let $\mathcal{S}$ be a set of non-negative bounded singular measures $\mu, \mu \neq 0$, on ∂D. Suppose that $\mathcal{S}$ satisfies the following conditions.*

(a) *For $\mu, \nu \in \mathcal{S}$, there exists $\lambda \in \mathcal{S}$ such that $\lambda \leq \mu \wedge \nu$, where $\mu \wedge \nu$ is the greatest lower bound of μ and ν,*

(b) *For every $\mu \in \mathcal{S}$ and a positive integer n, there exists $\lambda \in \mathcal{S}$ such that $n\lambda \leq \mu$.*

Let I be the ideal algebraically generated by singular functions $\psi_\mu, \mu \in \mathcal{S}$.

By Theorem 2.1 and Proposition 2.2, we have

COROLLARY 2.2. *Let f be a function in H^∞ which does not vanish on D. Let I be the ideal in H^∞ algebraically generated by functions $f^{1/n}, n = 1, 2, \ldots$. Then*

$\overline{I} = I(Z(I))$.

COROLLARY 2.3 [7, Theorem 4.2]. *Let E be a closed subset of $M(H^\infty) \setminus D$ such that $E \cap M(L^\infty) = \emptyset$. Let I be the ideal of functions in H^∞ which vanish on some open subsets U of $M(H^\infty) \setminus D$ such that $E \subset U$. Then $\overline{I} = I(Z(I))$.*

We also have the following.

COROLLARY 2.4. *Let I be a prime ideal in H^∞. Then $\overline{I} = I(Z(I))$.*

In [6], to prove that $I = I(Z(I))$ for a closed prime ideal I Gorkin and Mortini used the following formula given by Guillory and Sarason [9, pp.177-178]. Let R be an open subset of D such that $\partial R \cap D$ is a system of rectifiable curves. Then

$$\int_{\partial D} \frac{F}{u}\, dz = \int_{\partial R \cap D} \frac{F}{u}\, dz \tag{2.1}$$

for $F \in H^\infty$ and an inner function u satisfying $|u(z)| < \beta$ for $z \in R$ and $|u(z)| \geq \alpha$ for $z \in D \setminus R, 0 < \alpha < \beta < 1$. Formula (2.1) is used in several papers, see [8, 12, 13]. When u is not inner, equation (2.1) does not holds.

To prove Theorem 2.1, we need another formula similar to (2.1). The following theorem is interesting in its own right.

THEOREM 2.2. *Let $f \in H^\infty, \|f\|_\infty = 1$, and $0 < \varepsilon < 1/2 < \sigma < 1$. Let R be an open subset of D such that $\partial R \cap D$ is a system of rectifiable curves satisfying*

(i) $|f(z)| < \varepsilon$ *for* $z \in R$.

We assign the usual orientation on ∂R. Put $\Gamma = \partial R \cap D$. Let $h \in H^\infty$ such that $\|h\|_\infty = 1$,

(ii) $0 < 1/2 \leq |h(z)|$ *for* $z \in D \setminus R$,

(iii) $|h(e^{i\theta})| \geq \sigma$ *for almost every* $e^{i\theta} \in \partial D$ *with* $|f(e^{i\theta})| > \varepsilon$.

Then

$$\Big| \int_\Gamma \frac{fF}{h}\, dz - \int_{\partial D} fF\overline{h}\, dz \Big| \leq 4(\varepsilon + 1 - \sigma)\|F\|_1$$

for every $F \in H^\infty$, where $\|F\|_1 = \int_0^{2\pi} |F(e^{i\theta})|\, d\theta/2\pi$.

As an application of Theorem 2.2, we shall prove Theorem 2.1. Our theorems owe to the deep theorems due to Bourgain [2] and Suárez [18, 19].

Let $g(z) = (1 - z)/2$. Then g is an outer function and is not invertible in H^∞. Let $I = gH^\infty$ be the ideal generated by g. Then it is not difficult to see that for $h \in I$,

$$\Big\| h - hg\Big(\sum_{k=0}^{n-1} \Big(\frac{1+z}{2}\Big)^k\Big)\Big\|_\infty = \Big\| h - h\Big(1 - \Big(\frac{1+z}{2}\Big)^n\Big)\Big\|_\infty \to 0 \quad \text{as } n \to \infty.$$

Hence $\overline{I} = I(Z(I))$. One might ask whether $\overline{I} = I(Z(I))$ for an ideal I generated by a single outer function in H^∞ which is not invertible in H^∞. To answer this question, we need to recall Jensen's equality. For a point $\varphi \in M(H^\infty)$, there is a probability measure μ_φ on $M(L^\infty)$ such that $\int_{M(L^\infty)} f\, d\mu_\varphi = \varphi(f)$ for every $f \in H^\infty$. We denote by $supp\, \mu_\varphi$ the closed support set of μ_φ. Then the following Jensen inequality holds

$$\log|\varphi(f)| \leq \int_{M(L^\infty)} \log|f|\, d\mu_\varphi, \quad f \in H^\infty.$$

When it holds that

$$\log|\varphi(f)| = \int_{M(L^\infty)} \log|f|\, d\mu_\varphi,$$

we say that f satisfies Jensen's equality for $\varphi \in M(H^\infty)$. It is well known that every invertible function in H^∞ satisfies Jensen's equality for every point in $M(H^\infty)$, see [10, Chapter 10]. Our third theorem is

THEOREM 2.3. *Let f be an outer function in H^∞ which is not invertible in H^∞ Let $I = fH^\infty$ be the ideal generated by f. Then $\overline{I} = I(Z(I))$ if and only if f satisfies Jensen's equality for every point m in $M(H^\infty)$ with $m(f) \neq 0$.*

Axler and Shields [1, Proposition 5] showed that a function f in H^∞ satisfying $\mathrm{Re}\, f > 0$ on D satisfies Jensen's equality for every point in $M(H^\infty)$. For an inner function q, the function $q + 1$ satisfies this condition. Put $QA = H^\infty \cap \overline{H^\infty + C}$, where C is the space of continuous functions on ∂D and $\overline{H^\infty + C}$ is the set of complex conjugates of functions in $H^\infty + C$. In [20], Wolff proved that for every $f \in L^\infty$ there exists an outer function $h \in QA$ such that $hf \in H^\infty + C$. When $f \notin H^\infty + C$, the function h is not invertible in H^∞. So that there are many outer functions in QA which are not invertible in H^∞. In [17], Sarason proved that if $f \in H^\infty$, then $f \in QA$ if and only if $f_{|supp\, \mu_\varphi}$ is constant for every $\varphi \in M(H^\infty)\backslash D$. Hence QA outer functions satisfy Jensen's equality for every $\varphi \in M(H^\infty)$. We have following corollaries as applications of Theorem 2.3.

COROLLARY 2.5. *Let $I = fH^\infty$ be an ideal in H^∞ generated by a function f which is not invertible in H^∞ and $\mathrm{Re}\, f > 0$ on D. Then $\overline{I} = I(Z(I))$.*

COROLLARY 2.6. *Let $I = fH^\infty$ be an ideal in H^∞ generated by an outer function in QA which is not invertible in H^∞. Then $\overline{I} = I(Z(I))$.*

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